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On the slow motion of a solid submerged in a fluid with a surfactant surface film

R. SHAIL

Department of Mathematics, University of Surrey, Guildford, Surrey GU2 5XH, England

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Summary

This paper continues the work of Shail and Gooden [1–4] on the motion generated in a semi-infinite fluid by a singularity or submerged solid moving particle when the surface of the fluid is contaminated with a surfactant film. The fluid motion is assumed to be slow and quasi-steady, but the restriction to axially symmetric flows of earlier investigations is removed. The various linearised models of Shail and Gooden [3,4] governing the variation of film concentration are discussed, the constitutive properties of the film being expressed in terms of Boussinesq coefficients of surface shear and dilatational viscosities. The resulting film boundary conditions are applied to solve the non-axially symmetric problem of a Stokeslet placed in the bulk fluid with its axis parallel to the surface (assumed planar throughout the motion), and the results used to calculate approximate expressions for the resistive force on a particle which translates far from and parallel to the surface. A similar analysis is given for the case of a rotelet whose axis is parallel to the surface.

1. Introduction

In a series of papers [1–4] Shail and Gooden have investigated various aspects of the fluid motion generated when a submerged solid rotates or translates slowly in a semi-infinite or bounded viscous fluid whose surface is contaminated with a surfactant film. Papers [1] and [2] concern the rotation of a solid about an axis perpendicular to the surface, and in [3] and [4] the work is extended to the translation of the body in a direction perpendicular to the surface film. All the fluid motions considered so far have been axisymmetric, and it is the purpose of this paper to initiate the extension of the analysis to problems of a non-axially symmetric nature whch arise, for example, when a body translates along or rotates about an axis parallel to the surface film. As in [3] and [4] the bulk fluid motion is assumed to be sufficiently slow to permit the quasi-steady Stokes approximation to be made, and the constitutive properties of the surface film are described in terms of the Boussinesq coefficients of surface shear and dilatational viscosity, η and κ. The form of the dynamic boundary condition at the surfactant film is taken to be that proposed by Scriven [5], and the physical processes governing surface concentration of surfactant comprise surface diffusion, adsorbtion and desorbtion, and for a soluble surfactant bulk diffusion into the film from the substrate fluid.

In Section 2 the equations of fluid motion are formulated using cylindrical polar coordinates (ρ, ϕ, z). Suitable forms for the dynamic boundary conditions appropriate to the non-axially symmetric fluid motion are deduced, it being assumed as in [3] and [4] that the surface film remains plane and incident with $z = 0$ throughout the motion. To these dynamic conditions must be appended the conditions governing the film surface concentration (which is related via an equation of state to the film surface pressure), and forms for the various film processes, linearised about the equilibrium film state after the manner suggested by Levich [6] and used in [3] and [4], are given.

In Section 3 the various model conditions are applied to the problem of finding the motion generated in the bulk fluid by a Stokeslet placed with its axis parallel to the planar fluid surface. The technique of expressing the fluid velocity vector in terms of harmonic functions is exploited, and the solution is used in conjunction with Brenner's analysis [7] to derive approximate expressions for the drag on a body translating along a principal axis of resistance which is parallel to and at a large distance from the surfactant film. In Section 4 the same methods are applied to treat the problem of a rotelet placed in the bulk fluid with its axis parallel to the surface film. Again this solution and Brenner's technique lead to asymptotic expressions for the frictional couple on a body rotating about a principal axis of resistance which is parallel to the surface. In the concluding section some further avenues of research are outlined.

2. Basic equations

A semi-infinite expanse of viscous incompressible fluid, with coefficient of viscosity μ, occupies the region $z > 0$, where (ρ, ϕ, z) are cylindrical polar coordinates with z-axis drawn vertically downwards. The surface $z = 0$ is contaminated with a monomolecular surfactant film whose coefficients of surface shear and dilatational viscosity are η and κ, respectively. The bulk liquid motion is assumed to be generated by a singularity (Stokeslet or rotelet) or moving solid body, and the fluid velocity vector $\boldsymbol{u}(\rho, \phi, z)$ has cylindrical polar components (u, v, w). The motion is considered to be sufficiently slow for the quasi-steady Stokes creeping-motion approximation to be made, which requires that $Ua/\nu \ll 1$ and $Ua^2/\nu h \ll 1$, where ν is the kinematic viscosity of the bulk fluid, a a typical dimension of the moving solid, h a measure of its depth below the surface and U its speed. The linearised time-independent equations of motion and continuity then read

$$\frac{1}{\mu}\frac{\partial p}{\partial \rho} = \left(\nabla^2 - \frac{1}{\rho^2}\right)u - \frac{2}{\rho^2}\frac{\partial v}{\partial \phi}, \tag{1}$$

$$\frac{1}{\mu\rho}\frac{\partial p}{\partial \phi} = \left(\nabla^2 - \frac{1}{\rho^2}\right)v + \frac{2}{\rho^2}\frac{\partial u}{\partial \phi}, \tag{2}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial z} = \nabla^2 w, \tag{3}$$

and

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho u) + \frac{1}{\rho}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0, \tag{4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}, \tag{5}$$

and p is the dynamic pressure.

Following [3] and [4] the surfactant layer is assumed to remain plane and incident with $z=0$ throughout the motion, which implies that

$$w=0 \quad \text{on} \quad z=0, 0 \leqslant \rho < \infty. \tag{6}$$

Further all components of velocity and stress are required to tend to zero as $\rho^2 + z^2 \to \infty$, and at a solid boundary the usual no-slip condition is imposed on the fluid velocity vector.

Consider next the dynamic conditions at the surfactant film. The Scriven boundary conditions [5] on $z=0$, which express the balance of substrate and internal film stresses, can be written in the form

$$-\tau_{\rho z}\boldsymbol{\rho} - \tau_{\phi z}\boldsymbol{\phi} = -\nabla p_s + (\kappa+\eta)\nabla(\operatorname{div}\boldsymbol{u}) - \eta \operatorname{curl}\operatorname{curl}\boldsymbol{u}, \tag{7}$$

where the vector operators are two-dimensional operators in the plane $z=0$, $\tau_{\rho z}$ and $\tau_{\phi z}$ are bulk-fluid components of stress, p_s is the surface pressure, and $\boldsymbol{u} = (u(\rho, \phi, 0), v(\rho, \phi, 0), 0)$. Further, $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$ are unit vectors in the directions of ρ- and ϕ-increasing, and by virtue of (6), on $z=0$,

$$\tau_{\rho z} = \mu \frac{\partial u}{\partial z}, \qquad \tau_{\phi z} = \mu \frac{\partial v}{\partial z}.$$

Thus, the ρ- and ϕ-components of (7) are

$$-\mu\frac{\partial u}{\partial z} = -\frac{\partial p_s}{\partial \rho} + (\kappa+\eta)\frac{\partial}{\partial \rho}\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho u) + \frac{1}{\rho}\frac{\partial v}{\partial \phi}\right\} - \frac{\eta}{\rho^2}\frac{\partial}{\partial \phi}\left\{\frac{\partial}{\partial \rho}(\rho v) - \frac{\partial u}{\partial \phi}\right\}, \tag{8}$$

and

$$-\mu\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p_s}{\partial \phi} + (\kappa+\eta)\frac{1}{\rho^2}\frac{\partial}{\partial \phi}\left\{\frac{\partial}{\partial \rho}(\rho u) + \frac{\partial v}{\partial \phi}\right\} + \eta\frac{\partial}{\partial \rho}\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v) - \frac{1}{\rho}\frac{\partial u}{\partial \phi}\right\}, \tag{9}$$

on $z=0$. Using the equations of motion and continuity (1) to (4) it can be shown that on $z=0$,

$$\frac{\partial}{\partial \rho}\left\{\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho v) - \frac{1}{\rho}\frac{\partial u}{\partial \phi}\right\} = \frac{1}{\mu\rho}\frac{\partial p}{\partial \phi} + \frac{1}{\rho}\frac{\partial^2 w}{\partial \phi \partial z} - \frac{\partial^2 v}{\partial z^2}, \tag{10}$$

and

$$\frac{1}{\rho^2}\frac{\partial}{\partial \phi}\left\{\frac{\partial}{\partial \rho}(\rho v) - \frac{\partial u}{\partial \phi}\right\} = -\frac{1}{\mu}\frac{\partial p}{\partial \rho} - \frac{\partial^2 w}{\partial \rho \partial z} + \frac{\partial^2 u}{\partial z^2}. \tag{11}$$

Equations (10) and (11) together with (4) now enable the surface conditions (8) and (9) to be expressed conveniently as

$$\mu\frac{\partial u}{\partial z} = \frac{\partial}{\partial \rho}\left(p_s - \frac{\eta}{\mu}p\right) + \kappa\frac{\partial^2 w}{\partial \rho \partial z} + \eta\frac{\partial^2 u}{\partial z^2}, \tag{12}$$

$$\mu \frac{\partial v}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial \phi} \left(p_s - \frac{\eta}{\mu} p \right) + \frac{\kappa}{\rho} \frac{\partial^2 w}{\partial \phi \partial z} + \eta \frac{\partial^2 v}{\partial z^2}. \tag{13}$$

As noted in [3] it is not possible to impose simultaneously with (6), (12) and (13) the requirement of the vanishing of the normal stress τ_{zz} at the surface; the resulting normal stress imbalance is assumed to be compensated by surface tension effects (see [8] for a discussion of this point).

To (12) and (13) must be added the appropriate equations governing the surface concentration $n(\rho, \phi)$ in the film, and the relation of n to the surface pressure p_s via an equation of state. These conditions and their linearisation about an equilibrium state for slow motions are fully described in [3] where relevant references are given and we adopt the same notations in this work. Thus $n(\rho, \phi)$ is written as

$$n = n_0 + n', \; n' \ll n_0,$$

where n_0 is the equilibrium concentration. If the film is taken to be gaseous then the equation of state has the form

$$p_s = kTn, \tag{14}$$

where k is Boltzmann's constant and T is the (constant) temperature. Following the analysis of [3], the non-axially symmetric linearised film condition in the case where there is surface diffusion of the surfactant in the film, together with adsorbtion from and desorbtion to the bulk fluid, reads

$$n_0 \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho u) + \frac{1}{\rho} \frac{\partial v}{\partial \phi} \right\} = D_s \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial n'}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 n'}{\partial \phi^2} \right\} - \beta n' \tag{15}$$

on $z = 0$ (c.f. equation (17) of reference [3]). Here D_s is the coefficient of surface diffusion, and $\beta > 0$ the constant appearing in the adsorbtion-desorbtion flux (equation (15) of [3]). The left member of (15) may be simplified using the continuity equation (4), and if the surface concentration $n'(\rho, \phi)$ is regarded as the value on $z = 0$ of a harmonic function $n'(\rho, \phi, z)$ which is defined throughout $z \geqslant 0$ and vanishes at infinity, then (15) can be expressed as

$$n_0 \frac{\partial w}{\partial z} = D_s \frac{\partial^2 n'}{\partial z^2} + \beta n' \qquad \text{on} \qquad z = 0. \tag{16}$$

Relations (6), (12), (13) and (16), together with (14), now constitute a complete set of boundary conditions at the surfactant film $z = 0$ which, in conjunction with the no-slip condition at a moving solid, are sufficient for the determination of the velocity field throughout the fluid and the surface concentration of surfactant. It is readily verified that they reduce to the conditions used in previous papers [1–4] in the cases (i) axisymmetric rotational motions with $u = w = 0$ and $\partial/\partial\phi = 0$, and (ii) axisymmetric motions without swirl where $v = 0$ and $\partial/\partial\phi = 0$.

The remaining film process which we consider is diffusion of a soluble surfactant from the bulk fluid into the film. Let $c(\rho, \phi, z)$ denote the concentration of solute in the bulk fluid, and following [3] we write $c = c_0 + c'$, where c_0 is the stable equilibrium concentration. When the Peclet number $Ua/D_0 \ll 1$, where D_0 is the bulk diffusion coefficient, c' satisfies Laplace's equation

$$\nabla^2 c' = 0, \tag{17}$$

subject to an appropriate boundary condition at the surface of a moving solid, e.g. for an impermeable solid the normal derivative of c' vanishes at its surface. The equation of surface mass balance in the film takes the linearised form

$$n_0\left\{\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho u) + \frac{1}{\rho}\frac{\partial v}{\partial\phi}\right\} = \frac{D_0}{h_0}\frac{\partial n'}{\partial z} \qquad \text{on} \qquad z = 0, \tag{18}$$

where $n' = h_0 c'$, h_0 being the adsorbtion depth (see [9]), and the right-hand side of (18) is the diffusion flux into the film. (Note that although the surface concentration n' exists physically only on $z = 0$, a fictitious "surface concentration" $n'(\rho, \phi, z)$ may be defined throughout the fluid by the relation $n' = h_0 c'$.) Again (18) may be simplified using the equation of continuity, and (16), (18) may be written compendiously as

$$n_0\frac{\partial w}{\partial z} = D_s\frac{\partial^2 n'}{\partial z^2} - \frac{D_0}{h_0}\frac{\partial n'}{\partial z} + \beta n' \qquad \text{on} \qquad z = 0, \tag{19}$$

a form which subsumes, for appropriate values of D_s, D_0 and β, each of the film processes examined.

3. The Stokeslet problem and drag formulae

As a preliminary to estimating the drag on a particle which translates parallel to the surface film, we consider the motion generated by a Stokeslet of strength f, placed at the point $(0, 0, h)$ in the bulk fluid with its axis oriented parallel to the Cartesian x-axis. In an unbounded fluid the Stokeslet produces a velocity field $\boldsymbol{v}_0(\boldsymbol{r}, h)$, with cylindrical polar components

$$u_0 = f\left\{\frac{2}{R_1} - \frac{(z-h)^2}{R_1^3}\right\}\cos\phi,$$

$$v_0 = -\frac{f\sin\phi}{R_1},$$

$$w_0 = \frac{f\rho(z-h)}{R_1^3}\cos\phi, \tag{20}$$

the associated pressure field being

$$p_0(\boldsymbol{r}, h) = \frac{2\mu f\rho\cos\phi}{R_1^3}, \tag{21}$$

where $R_1 = \{\rho^2 + (z-h)^2\}^{1/2}$. In the semi-infinite fluid situation a suitable solution $(\boldsymbol{v}, p)$

of (1) to (4) can be constructed in terms of harmonic functions $X = \chi(\rho, z)\cos\phi$ and $\Theta = \theta(\rho, z)\sin\phi$, which vanish as $\rho^2 + z^2 \to \infty$, as

$$\boldsymbol{v} = \boldsymbol{v}_0(\boldsymbol{r}, h) + \boldsymbol{v}_0(\boldsymbol{r}, -h) + z\nabla\left(\frac{\partial X}{\partial z}\right) - \frac{\partial X}{\partial z}\boldsymbol{z} + \nabla X + \mathrm{curl}(\Theta\boldsymbol{z}), \tag{22}$$

and

$$p = p_0(\boldsymbol{r}, h) + p_0(\boldsymbol{r}, -h) + 2\mu\frac{\partial^2 X}{\partial z^2}, \tag{23}$$

thereby satisfying condition (6) at the outset. (In (22) $\boldsymbol{z}$ is a unit vector parallel to the z-axis, and the second term is an image Stokeslet.) Writing the cylindrical polar components of velocity (u, v, w) in the forms

$$u = u_1(\rho, z)\cos\phi, \qquad v = v_1(\rho, z)\sin\phi, \qquad w = w_1(\rho, z)\cos\phi,$$

the pressure p as $p = p_1(\rho, z)\cos\phi$, and the non-equilibrium part of the surface pressure p_s as

$$p_s = p_{s1}(\rho)\cos\phi = kTn_1(\rho)\cos\phi\,^{\dagger},$$

then from (22) and (23),

$$u_1 = 2f\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - f\left[\frac{(z-h)^2}{R_1^3} + \frac{(z+h)^2}{R_2^3}\right] + z\frac{\partial^2\chi}{\partial\rho\partial z} + \frac{\partial\chi}{\partial\rho} + \frac{\theta}{\rho}, \tag{24}$$

$$v_1 = -f\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - \frac{z}{\rho}\frac{\partial\chi}{\partial z} - \frac{\chi}{\rho} - \frac{\partial\theta}{\partial\rho}, \tag{25}$$

$$w_1 = f\rho\left(\frac{z-h}{R_1^3} + \frac{z+h}{R_2^3}\right) + z\frac{\partial^2\chi}{\partial z^2}, \tag{26}$$

and

$$p_1 = 2\mu f\rho\left(\frac{1}{R_1^3} + \frac{1}{R_2^3}\right) + 2\mu\frac{\partial^2\chi}{\partial z^2}, \tag{27}$$

where $R_2 = \{\rho^2 + (z+h)^2\}^{1/2}$. Further, the boundary conditions (12) and (13) become

$$\mu\frac{\partial u_1}{\partial z} = \frac{\partial\pi_1}{\partial\rho} + \kappa\frac{\partial^2 w_1}{\partial\rho\partial z} + \eta\frac{\partial^2 u_1}{\partial z^2}, \tag{28}$$

$$\mu\frac{\partial v_1}{\partial z} = -\frac{\pi_1}{\rho} - \frac{\kappa}{\rho}\frac{\partial w_1}{\partial z} + \eta\frac{\partial^2 v_1}{\partial z^2}, \tag{29}$$

† Here $n'(\rho, \phi) = n_1(\rho)\cos\phi$, and $n_1(\rho)$ can be regarded as the value on $z = 0$ of a function $n_1(\rho, z)$ in $z \geqslant 0$ such that $n_1(\rho, z)\cos\phi$ is harmonic.

on $z=0$, where

$$\pi_1 = p_{s1}(\rho) - \frac{\eta}{\mu} p_1(\rho, 0)$$

$$= kTn_1(\rho) - 2\eta\left\{\frac{2f\rho}{R^3} + \left(\frac{\partial^2\chi}{\partial z^2}\right)_{z=0}\right\}, \tag{30}$$

with $R = (\rho^2 + h^2)^{1/2}$.

Using (24) to (27) in (28) and (29), a straight-forward calculation yields the relations

$$\mu\left(2\frac{\partial^2\chi}{\partial\rho\partial z} + \frac{1}{\rho}\frac{\partial\theta}{\partial z}\right) = \frac{\partial\pi_1}{\partial\rho} + \kappa\left(\frac{\partial^3\chi}{\partial\rho\partial z^2} - \frac{4f}{R^3} + \frac{30fh^2\rho^2}{R^7}\right)$$

$$+\eta\left(3\frac{\partial^3\chi}{\partial\rho\partial z^2} + \frac{1}{\rho}\frac{\partial^2\theta}{\partial z^2} - \frac{8f}{R^3} + \frac{42fh^2}{R^5} - \frac{30fh^4}{R^7}\right), \tag{31}$$

and

$$\mu\left(\frac{2}{\rho}\frac{\partial\chi}{\partial z} + \frac{\partial^2\theta}{\partial\rho\partial z}\right) = \frac{\pi_1}{\rho} + \kappa\left(\frac{1}{\rho}\frac{\partial^2\chi}{\partial z^2} + \frac{2f}{R^3} - \frac{6fh^2}{R^5}\right)$$

$$+\eta\left(\frac{3}{\rho}\frac{\partial^2\chi}{\partial z^2} + \frac{\partial^3\theta}{\partial\rho\partial z^2} - \frac{2f}{R^3} + \frac{6fh^2}{R^5}\right), \tag{32}$$

where all the partial derivatives are evaluated on $z=0$. Adding and subtracting (31) and (32), and introducing the operators $\mathcal{D}_\pm = (\partial/\partial\rho) \pm \rho^{-1}$, the boundary conditions can be reduced to the concise forms

$$\mathcal{D}_\pm\left\{(\kappa + 3\eta)\frac{\partial^2\chi}{\partial z^2} \pm \eta\frac{\partial^2\theta}{\partial z^2} - 2\mu\frac{\partial\chi}{\partial z} \mp \mu\frac{\partial\theta}{\partial z} + \pi_1\right\} = f_\pm(\rho), \tag{33}$$

where

$$f_+(\rho) = 2f\left\{\frac{\kappa + 5\eta}{R^3} - 12(\kappa + 12\eta)\frac{h^2}{R^5} + 15(\kappa + \eta)\frac{h^4}{R^7}\right\}, \tag{34}$$

$$f_-(\rho) = 6f(\kappa + \eta)\left(\frac{1}{R^3} - \frac{6h^2}{R^5} + \frac{5h^4}{R^7}\right). \tag{35}$$

Suitable representations for χ, θ, vanishing as $\rho^2 + z^2 \to \infty$, are

$$\chi(\rho, z) = \int_0^\infty A(s) J_1(s\rho)\, e^{-sz} ds, \tag{36}$$

$$\theta(\rho, z) = \int_0^\infty B(s) J_1(s\rho)\, e^{-sz} ds, \tag{37}$$

and π_1 has the Hankel transform

$$\pi_1 = \int_0^\infty C(s) J_1(s\rho)\mathrm{d}s. \tag{38}$$

Substitution of (36) to (38) into $(33)_+$, and use of the fact that

$$\mathcal{D}_+ J_1(s\rho) = sJ_0(s\rho)$$

produce

$$\int_0^\infty s\left[\{(\kappa + 3\eta)s^2 + 2\mu s\}A + s(\eta s + \mu)B + C\right] J_0(s\rho)\mathrm{d}s = f_+(\rho), 0 \leqslant \rho < \infty; \tag{39}$$

thus, invoking the Hankel transform theorem,

$$s\{(\kappa + 3\eta)s + 2\mu\}A + s(\eta s + \mu)B + C = \int_0^\infty \rho f_+(\rho) J_0(s\rho)\mathrm{d}\rho$$

$$= 2f\{-(\kappa + 5\eta) + hs\}s\,\mathrm{e}^{-hs}, \qquad 0 \leqslant s < \infty, \tag{40}$$

using the various integrals collected in the Appendix. In a similar manner $(33)_-$, the relation $\mathcal{D}_- J_1(s\rho) = -sJ_2(s\rho)$, and the Hankel inversion theorem give

$$s\{(\kappa + 3\eta)s + 2\mu\}A - s(\eta s + \mu)B + C = -\int_0^\infty \rho f_-(\rho) J_2(s\rho)\mathrm{d}\rho$$

$$= 2f(\kappa + \eta)(hs - 1)s\,\mathrm{e}^{-hs}, 0 \leqslant s < \infty. \tag{41}$$

It now follows immediately from (39) and (40) that

$$B(s) = -\frac{4f\eta\,\mathrm{e}^{-hs}}{\eta s + \mu}, \tag{42}$$

with

$$s\{(\kappa + 3\eta)s + 2\mu\}A + C = 2f\{(\kappa + \eta)hs - (\kappa + 3\eta)\}s\,\mathrm{e}^{-hs}. \tag{43}$$

To supply a further relation between A and C we next apply (19). Writing

$$n_1(\rho, z) = \int_0^\infty D(s) J_1(s\rho)\,\mathrm{e}^{-sz}\mathrm{d}s, \tag{44}$$

then (30), (36) and (44) show that

$$C = kTs^{-1}D - 2\eta sA - 4\eta f\,\mathrm{e}^{-sh}, \tag{45}$$

where integral (A5) has been used. Also from (26) and (19),

$$n_0\left(\frac{2f\rho}{R^3} - \frac{6fh^2\rho}{R^5} + \frac{\partial^2\chi}{\partial z^2}\right) = D_s\frac{\partial^2 n_1}{\partial z^2} - \frac{D_0}{h_0}\frac{\partial n_1}{\partial z} + \beta n_1, \tag{46}$$

on $z = 0$. Thus (36), (44) and appropriate integrals from the Appendix give

$$n_0\{2f(1-hs)\,\mathrm{e}^{-hs} + sA\} = s^{-1}\Delta(s)D, \tag{47}$$

where $\Delta(s) = D_0 s^2 + (D_0/h_0)s + \beta$. Solving (43), (45) and (47) it is found that

$$A(s) = 2f\,\mathrm{e}^{-hs}(hs-1)\frac{\{n_0kT + (\kappa+\eta)\Delta\}}{n_0kTs + \Delta((\kappa+\eta)s + 2\mu)}, \tag{48}$$

$$D(s) = \frac{4n_0 f\mu s(1-hs)\,\mathrm{e}^{-hs}}{n_0kTs + \Delta((\eta+\kappa)s + 2\mu)}, \tag{49}$$

and the velocity, pressure and surface concentration are now fully determined.

We now turn to the application of this Stokeslet solution to the derivation of approximate expressions for the drag on a particle which translates without rotation with velocity $V\boldsymbol{i}$ parallel to the x-axis. The axis of translation is assumed to be a principal axis of resistance of the particle, and $F\boldsymbol{i}$, $F_\infty \boldsymbol{i}$ denote the viscous drag forces for the semi-infinite and an everywhere infinite bulk fluid. If a denotes a typical dimension and h measures the depth below the surface film of a suitable centre Q of the body, then Brenner [7] has provided an approximation to F correct, in general, to $O(a^2/h^2)$. † Specifically, for translating bodies of the proposed symmetry and writing $\varepsilon = a/h$, the ratio F/F_∞ has the form

$$\frac{F}{F_\infty} = \frac{1}{1 - K(F_\infty/6\pi\mu Ua)\varepsilon + O(\varepsilon^3)}, \tag{50}$$

where K is a constant depending only on the presence of the surface film and not on the detailed particle geometry. Brenner's derivation of (50) by the method of reflexions requires that the boundary conditions at the surface of the fluid are linear and homogeneous, which is the case in the present work; an alternative approach to (50) and the estimation of the error can be given using the integral-equation method proposed by Williams [10].

In order to calculate K we follow the prescription given by Brenner [7], and Happel and Brenner [11]. First the Stokeslet strength f is identified with $F_\infty/6\pi\mu$; then if $\boldsymbol{v}_1(Q)$ denotes the regular part of (22), i.e. $\boldsymbol{v} - \boldsymbol{v}_0(\boldsymbol{r}, h)$, evaluated at the centre $Q(0, 0, h)$ of the body, we have that

$$K = -\frac{6\pi\mu h}{F_\infty}(\boldsymbol{v}_1(Q)\cdot\boldsymbol{i}). \tag{51}$$

Thus, from (22) and the results of the Stokeslet analysis, K is found as

$$K = -\frac{3}{8} + \frac{3h}{4}\int_0^\infty s\,\mathrm{e}^{-2hs}\left[\frac{(hs-1)^2\{(\kappa+\eta)\Delta + n_0kT\}}{n_0kTs + \Delta((\kappa+\eta)s + 2\mu)} + \frac{2\eta}{\eta s + \mu}\right]\mathrm{d}s. \tag{52}$$

† An exception to this order arises in certain problems involving bulk diffusion into the surface film, and has been exemplified in [3] and [4].

Since the error estimate in the denominator of (50) is $O(\varepsilon^3)$, the asymptotic expansion of (52), correct to $O(\varepsilon)$, is required for use in (50).

We next consider the drag formulae provided by (50) and (52) in various representative asymptotic situations. Suppose first that the surfactant is insoluble with surface diffusion as the controlling film process; then $\Delta(s) = D_s s^2$ and from (52)

$$K = -\tfrac{3}{8} + \tfrac{3}{4}K_1, \tag{53}$$

where

$$K_1 = \int_0^\infty (t-1)^2 \frac{\left(1 + N_2\varepsilon^2 t^2\right)}{1 + N_1\varepsilon t + N_2\varepsilon^2 t^2} e^{-2t}\mathrm{d}t + 2N_0\varepsilon \int_0^\infty \frac{t}{1 + N_0\varepsilon t} e^{-2t}\mathrm{d}t. \tag{54}$$

In (54), $N_0 = \eta/\mu a$, $N_1 = 2\mu D_s/n_0 kTa$ and $N_2 = (\eta + \kappa)D_s/n_0 kTa^2$ are three dimensionless parameters; N_0 appears in the rotation problems of [1] and [2] (where it is denoted by λ), whereas N_1 and N_2 were introduced in [3], to which the reader is referred for some typical values. When N_0, N_1, N_2 are all of $O(1)$, the expansion of (54) for $\varepsilon \ll 1$ yields

$$K_1 = \tfrac{1}{4} + \tfrac{1}{2}N_0\varepsilon - \tfrac{1}{8}N_1\varepsilon + O(\varepsilon^2), \tag{55}$$

with

$$K = -\tfrac{3}{16} - \tfrac{3}{32}(N_1 - 4N_0)\varepsilon + O(\varepsilon^2). \tag{56}$$

Thus, from (50)

$$\frac{F}{F_\infty} = 1 - \tfrac{3}{16}\Phi\varepsilon + \tfrac{3}{32}\Phi\left(\tfrac{3}{8}\Phi - N_1 + 4N_0\right)\varepsilon^2 + O(\varepsilon^3), \tag{57}$$

where $\Phi = F_\infty/6\pi\mu Ua$. For a spherical particle $\Phi = 1$, whereas for a disk-shaped particle moving normal to its plane, $\Phi = 8/3\pi$. Two limits of (54) are worth noting; letting $N_0 \to 0$ and $N_1 \to \infty$ with N_2 fixed, we find that $K = -3/8$, the value appropriate to an uncontaminated free surface [7]. Letting $N_0 \to \infty$, $N_2 \to \infty$ with N_1 fixed, then $K_1 = 15/16$ and $K = 9/16$, again in agreement with the result quoted in [7] for a rigid plane bounding surface.

We turn now to the absorbtion-desorbtion film process, for which $D_s = D_0 = 0$ with $\Delta(s) = \beta$. Then

$$K = -\tfrac{3}{8} + \tfrac{3}{4}K_2, \tag{58}$$

where

$$K_2 = \varepsilon N_3 \int_0^\infty \frac{t(t-1)^2}{1 + \varepsilon N_3 t} e^{-2t}\mathrm{d}t + 2N_0\varepsilon \int_0^\infty \frac{t}{1 + N_0\varepsilon t} e^{-2t}\mathrm{d}t, \tag{59}$$

with $N_3 = (n_0 kT\beta^{-1} + \eta + \kappa)/2\mu a$, a dimensionless group introduced in [3]. We first examine the situation in which N_0, $N_3 \gg 1$, $\varepsilon \ll 1$, with $\Lambda_1 = N_3\varepsilon$ and $\Lambda_2 = N_0\varepsilon$ both of order unity. Then K_2 in (59) is $O(1)$, and the various integrals can be evaluated in terms of the exponential integral $E_1(x)$ (see [12]). It is found that

$$K_2 = \frac{5}{4} + \frac{3}{4\Lambda_1} + \frac{1}{2\Lambda_1^2} - \frac{1}{\Lambda_1}\left(\frac{1}{\Lambda_1}+1\right)^2 e^{2/\Lambda_1}E_1(2/\Lambda_1) - \frac{2}{\Lambda_2}e^{2/\Lambda_2}E_1(2/\Lambda_2), \tag{60}$$

whence

$$K = \tfrac{9}{16}\left(1+\frac{1}{\Lambda_1}\right) + \frac{3}{8\Lambda_1^2} - \frac{3}{4\Lambda_1}\left(\frac{1}{\Lambda_1}+1\right)^2 e^{2/\Lambda_1}E_1(2/\Lambda_1) - \frac{3}{2\Lambda_2}e^{2/\Lambda_2}E_1(2/\Lambda_2). \tag{61}$$

Since $K = O(1)$, (50) shows that

$$\frac{F}{F_\infty} = 1 + K\Phi\varepsilon + K^2\Phi^2\varepsilon^2 + O(\varepsilon^3), \tag{62}$$

with K recorded in (61).

Suppose now that N_0, $N_3 = O(1)$; expansion of (59) for $\varepsilon \ll 1$, together with (58), now give

$$K = -\tfrac{3}{8}\{1 - \varepsilon(N_0 + \tfrac{1}{4}N_3)\} + O(\varepsilon^2), \tag{63}$$

and F/F_∞ follows as

$$\frac{F}{F_\infty} = 1 - \tfrac{3}{8}\varepsilon\Phi + \tfrac{3}{8}\Phi(\tfrac{3}{8}\Phi + N_0 + \tfrac{1}{4}N_3)\varepsilon^2 + O(\varepsilon^3). \tag{64}$$

Again the limit N_0, $N_3 \to 0$ reproduces the correct result for a free clean surface.

To complete the discussion of drag forces we briefly treat the case of bulk diffusion into the film for which $D_s = \beta = 0$ and $\Delta(s) = (D_0/h_0)s$. Then

$$K = -\tfrac{3}{8} + \tfrac{3}{4}K_3, \tag{65}$$

where, from (52),

$$K_3 = \int_0^\infty (t-1)^2 \frac{(N_4 + N_5\varepsilon t)}{1 + N_4 + N_5\varepsilon t} e^{-2t}dt + 2N_0\varepsilon\int_0^\infty \frac{t}{1+N_0\varepsilon t}e^{-2t}dt. \tag{66}$$

As in [3], the dimensionless quantities N_4, N_5 are defined by $N_4 = h_0 n_0 kT/2\mu D_0$ and $N_5 = (\eta + \kappa)/2\mu a$. By way of illustration, suppose that N_4 and N_5 are both $O(1)$; then from (65) and (66) we find that

$$K = -\frac{3}{16}\left\{\frac{N_4+2}{N_4+1} - \frac{N_5\varepsilon}{2(N_4+1)^2} + 2N_0\varepsilon\right\} + O(\varepsilon^2),$$

and the drag ratio F/F_∞ is

$$\frac{F}{F_\infty} = 1 - \frac{3(N_4+2)\Phi}{16(N_4+1)}\varepsilon + \tfrac{3}{16}\Phi\left\{\frac{N_5}{2(N_4+1)^2} + \frac{3(N_4+2)^2}{16(N_4+1)^2}\Phi + 2N_0\right\}\varepsilon^2 + O(\varepsilon^3). \tag{67}$$

It was pointed out in [3] that only the first two terms in a formula such as (67) follow rigorously from the method of reflexions, since the boundary condition on the solute concentration at the moving particle surface is not invoked. However the work of [4] indicates that the corresponding $O(\varepsilon^2)$-term in axisymmetric translation problems is valid for an impermeable solid, and we conjecture that this is the case in (67).

4. The rotelet problem and resistive torques

In this section the Stokeslet singularity of the preceding analysis is replaced by a rotelet of strength γ situated at $(0, 0, h)$ with axis parallel to the x-axis. Thus in an everywhere-infinite fluid the rotelet produces a velocity field $\boldsymbol{v}_0^*(\boldsymbol{r}, h)$ with cylindrical components

$$u_0^* = -\gamma\frac{(z-h)}{R_1^3}\sin\phi,\ v_0^* = -\gamma\frac{(z-h)}{R_1^3}\cos\phi,\ w_0^* = \frac{\gamma\rho}{R_1^3}\sin\phi, \tag{68}$$

the pressure being constant throughout the fluid. For the semi-infinite fluid configuration a suitable solution $(\boldsymbol{v}, p)$ of (1) to (4) is again expressible in terms of two harmonic functions $\mathrm{X} = \chi(\rho, z)\sin\phi$ and $\Theta = \theta(\rho, z)\cos\phi$, which vanish as $\rho^2 + z^2 \to \infty$, and an image rotelet as

$$\boldsymbol{v} = \boldsymbol{v}_0^*(\boldsymbol{r}, h) - \boldsymbol{v}_0^*(\boldsymbol{r}, -h) + z\nabla\left(\frac{\partial \mathrm{X}}{\partial z}\right) - \frac{\partial \mathrm{X}}{\partial z}\boldsymbol{z} + \nabla \mathrm{X} + \operatorname{curl}\Theta\boldsymbol{z}, \tag{69}$$

and

$$p = 2\mu\frac{\partial^2 \mathrm{X}}{\partial z^2}, \tag{70}$$

where the pressure at infinity is taken as zero. It is evident, using (68), that (69) satisfies identically the requirement that $w = 0$ on $z = 0$.

Writing the cylindrical components of the velocity field as $u = u_1(\rho, z)\sin\phi$, $v = v_1(\rho, z)\cos\phi$, $w = w_1(\rho, z)\sin\phi$, and the bulk and surface pressure fields as

$$p = p_1(\rho, z)\sin\phi,\ p_s = p_{s1}(\rho)\sin\phi = kTn_1(\rho)\sin\phi,$$

then from (69) and (70),

$$u_1 = \gamma\left(\frac{z+h}{R_2^3} - \frac{z-h}{R_1^3}\right) + z\frac{\partial^2\chi}{\partial\rho\partial z} + \frac{\partial\chi}{\partial\rho} - \frac{\theta}{\rho}, \tag{71}$$

$$v_1 = \gamma\left(\frac{z+h}{R_2^3} - \frac{z-h}{R_1^3}\right) + \frac{z}{\rho}\frac{\partial\chi}{\partial z} + \frac{\chi}{\rho} - \frac{\partial\theta}{\partial\rho}, \tag{72}$$

$$w_1 = \gamma\rho\left(\frac{1}{R_1^3} - \frac{1}{R_2^3}\right) + z\frac{\partial^2\chi}{\partial z^2}, \tag{73}$$

and

$$p_1 = 2\mu \frac{\partial^2 \chi}{\partial z^2}. \tag{74}$$

Further, the surfactant boundary conditions (12) and (13) are expressed as

$$\mu \frac{\partial u_1}{\partial z} = \frac{\partial \pi_1}{\partial \rho} + \kappa \frac{\partial^2 w_1}{\partial \rho \partial z} + \eta \frac{\partial^2 u_1}{\partial z^2}, \tag{75}$$

and

$$\mu \frac{\partial v_1}{\partial z} = \frac{\pi_1}{\rho} + \frac{\kappa}{\rho} \frac{\partial w_1}{\partial z} + \eta \frac{\partial^2 v_1}{\partial z^2} \tag{76}$$

on $z = 0$, where

$$\pi_1 = kTn_1(\rho) - 2\eta \frac{\partial^2 \chi}{\partial z^2}. \tag{77}$$

After some calculation, the forms corresponding to (33), (34) and (35) are found as

$$\mathfrak{D}_{\pm}\left\{(\kappa + 3\eta)\frac{\partial^2 \chi}{\partial z^2} \mp \eta \frac{\partial^2 \theta}{\partial z^2} - 2\mu \frac{\partial \chi}{\partial z} \pm \mu \frac{\partial \theta}{\partial z} + \pi_1\right\} = g_{\pm}(\rho), \tag*{(78)$_\pm$}$$

where

$$g_+(\rho) = 6\gamma(\kappa + 2\eta)h\left(\frac{3}{R^5} - \frac{5h^2}{R^7}\right), \tag{79}$$

$$g_-(\rho) = 30\gamma\kappa h\left(\frac{1}{R^5} - \frac{h^2}{R^7}\right). \tag{80}$$

To proceed with the computation of χ and θ the Hankel transform representations (36) through (38) and (44) are again introduced. Application of $(78)_{\pm}$ and (16) then yield, in a similar manner to that of Section 3, the results

$$A(s) = -\frac{2\gamma s\,\mathrm{e}^{-hs}\{n_0kT + (\kappa + \eta)\Delta\}}{n_0kTs + \Delta((\kappa + \eta)s + 2\mu)}, \tag{81}$$

$$B(s) = \frac{2\gamma\eta s\,\mathrm{e}^{-hs}}{\eta s + \mu}, \tag{82}$$

and

$$D(s) = \frac{4n_0\gamma\mu s^2\,\mathrm{e}^{-hs}}{n_0kTs + \Delta((\kappa + \eta)s + 2\mu)}, \tag{83}$$

whereby the velocity, pressure and surface concentration are fully determined.

The rotelet solution is now applied to the estimation of the viscous torque on a solid of appropriate symmetry which rotates with angular speed Ω about an axis parallel to the x-axis. As in Section 3, a denotes a typical dimension of the rotating solid, and h is the depth below the surface of a suitable centre Q on the axis of rotation; further $-T\boldsymbol{i}$ and $-T_\infty\boldsymbol{i}$ denote the resistive torques acting on the solid for the semi-infinite and everywhere-infinite fluid cases. When $\varepsilon = a/h \ll 1$, Brenner [7,11] has shown that in general

$$\frac{T}{T_\infty} = \frac{1}{1 - K\left(T_\infty/8\pi\mu\Omega a^3\right)\varepsilon + O(\varepsilon^5)}, \tag{84}$$

where K again depends only on the presence of the surface film and not on body geometry. (For bulk-diffusion flows the $O(\varepsilon^5)$ error estimate is subject to the same caveat as in Section 3.) To derive an expression for K from the rotelet solution we first identify the rotelet strength γ with $T_\infty/8\pi\mu$. Writing $\boldsymbol{v}_1$ for the regular part of (69), i.e. $\boldsymbol{v} - \boldsymbol{v}_0^*(\boldsymbol{r}, h)$, and setting $\boldsymbol{w}_1 = \frac{1}{2}(\text{curl } \boldsymbol{v}_1)_Q$, then following Brenner's analysis it can be shown that

$$\boldsymbol{\omega}_1 = -\omega_1 \boldsymbol{i},$$

and

$$K = 8\pi\mu h^3\omega_1/T_\infty$$
$$= -\tfrac{1}{16} + h^3\int_0^\infty s^3\,\mathrm{e}^{-2hs}\left\{\frac{n_0kT + \Delta(\kappa+\eta)}{n_0kT_s + \Delta((\kappa+\eta)s + 2\mu)} + \frac{\eta}{2(\eta s + \mu)}\right\}\mathrm{d}s. \tag{85}$$

Thus, bearing in mind the error estimate in (84), the asymptotic expansion of (85) correct to $O(\varepsilon)$ is required.

We complete this section by listing T/T_∞ for the various film processes and representative asymptotic limits. In these results $\Psi = T_\infty/8\pi\mu\Omega a^3$, with $\Psi = 1$ for a rotating spherical particle.

(i) Surface diffusion model

The appropriate form for K is

$$K = -\tfrac{1}{16} + \int_0^\infty t^2\,\mathrm{e}^{-2t}\frac{\left(1 + N_2\varepsilon^2t^2\right)}{1 + N_1\varepsilon t + N_2\varepsilon^2t^2}\mathrm{d}t + \tfrac{1}{2}\varepsilon N_0\int_0^\infty \frac{t^3}{1 + N_0\varepsilon t}\mathrm{e}^{-2t}\mathrm{d}t. \tag{86}$$

When N_0, N_1, N_2 are all $O(1)$, (84) and (86) give

$$\frac{T}{T_\infty} = 1 - \tfrac{3}{16}\Psi\varepsilon^3 + \tfrac{3}{16}\Psi(N_0 - 2N_1)\varepsilon^4 + O(\varepsilon^5). \tag{87}$$

Letting $N_0 \to 0$ and $N_1 \to \infty$ with N_2 fixed in (86), we find that $K = -\frac{1}{16}$ for an uncontaminated free surface, whereas the limits N_0, $N_2 \to \infty$ with N_1 fixed gave $K = 5/16$

for a rigid bounding plane. These values may be confirmed from results given in [8].

(ii) Desorbtion / adsorbtion model

For this film process,

$$K = -\tfrac{1}{16} + \varepsilon\int_0^{\infty} t^3\,\mathrm{e}^{-2t}\left\{\frac{N_3}{1+\varepsilon N_3 t} + \frac{N_0}{2(1+\varepsilon N_0 t)}\right\}\mathrm{d}t. \tag{88}$$

When N_0, $N_3 \gg 1$ with $\Lambda_1 = N_3\varepsilon$ and $\Lambda_2 = N_0\varepsilon$ both of order unity, then

$$K = \tfrac{5}{16} - \tfrac{1}{8}\left(\frac{2}{\Lambda_1} + \frac{1}{\Lambda_2}\right) + \frac{1}{4}\left(\frac{2}{\Lambda_1^2} + \frac{1}{\Lambda_2^2}\right) - \frac{1}{\Lambda_1^2}\mathrm{e}^{2/\Lambda_1}E_1(2/\Lambda_1) - \frac{1}{2\Lambda_2^2}\mathrm{e}^{2/\Lambda_2}E_1(2/\Lambda_2), \tag{89}$$

with

$$\frac{T}{T_\infty} = 1 + K\Psi\varepsilon^3 + O(\varepsilon^5). \tag{90}$$

However, when N_0, $N_3 = O(1)$, expansion of (88) for $\varepsilon \ll 1$ gives

$$K = -\tfrac{1}{16} + \tfrac{3}{16}(N_0 + 2N_3)\varepsilon + O(\varepsilon^2), \tag{91}$$

with

$$\frac{T}{T_\infty} = 1 - \tfrac{1}{16}\Psi\varepsilon^3 + \tfrac{3}{16}\Psi(N_0 + 2N_3)\varepsilon^4 + O(\varepsilon^5). \tag{92}$$

(iii) Bulk-diffusion model

In this case

$$K = -\tfrac{1}{16} + \int_0^{\infty} t^2\,\mathrm{e}^{-2t}\left\{\frac{N_4 + N_5\varepsilon t}{1 + N_4 + N_5\varepsilon t} + \frac{N_0\varepsilon t}{2(1+N_0\varepsilon t)}\right\}\mathrm{d}t; \tag{93}$$

thus for N_0, N_4, N_5 all $O(1)$,

$$K = \frac{3N_4 - 1}{16(N_4+1)} + \frac{3}{16}\left\{N_0 + \frac{2N_5}{(N_4+1)^2}\right\}\varepsilon + O(\varepsilon^2), \tag{94}$$

and

$$\frac{T}{T_\infty} = 1 + \frac{(3N_4 - 1)}{16(N_4+1)}\Psi\varepsilon^3 + \tfrac{3}{16}\Psi\left\{N_0 + \frac{2N_5}{(N_4+1)^2}\right\}\varepsilon^4 + O(\varepsilon^5). \tag{95}$$

Again the $O(\varepsilon^4)$-term in (95) is conjectured to be valid for a rotating impermeable solid.

5. Conclusion

In this paper the general non-axially symmetric problem has been formulated for a submerged body which translates or rotates in a semi-infinite viscous fluid whose surface is contaminated with a surfactant film. The bulk-fluid motion is assumed to be sufficiently slow for the quasi-steady Stokes approximation to be used, and the dynamic boundary conditions which couple the motion of the film to that of the substrate are those due to Scriven [5]. In order to avoid a moving-boundary problem and to render the analysis tractable it has been assumed that any vertical motion of the surface can be neglected (suppressed by a sufficiently large surface tension, say), and that throughout the motion the film remains incident with a fixed horizontal plane. This approximation has been widely used, although recently Berdan and Leal [13] have examined methods of allowing for surface deformation. It may prove possible at a future date to employ their ideas in the surfactant case, and consequently eliminate the normal-stress imbalance which occur at the surface in the present model.

The Stokeslet and rotelet solutions represent a first application of the general non-axially symmetric formulation. These solutions can be regarded as providing elements of various Green's tensors, and are of interest in their own right (see [14] for some related considerations for the two-dimensional Laplace equation). However, when used in conjunction with the analysis of Brenner [7], they yield asymptotic estimates of the drag forces and torques on translating and rotating bodies which are far from the surface in the sense that $a/h \ll 1$. For the various film processes considered, no fewer than six dimensionless quantities, $N_0, \dots, N_5$, arise in the text, and depending on their orders in terms of ε, a large number of different asymptotic expressions for drag and torque can be envisaged; Sections 3 and 4 contain a representative but by no means exhaustive selection. In order to assess the accuracy of these it is desirable to have exact solutions for specific solids available for comparison. The provision of such solutions is a task to be undertaken, but based on the work in [4] concerning axisymmetric translation problems, percentage errors of less than 10% can be expected with $\varepsilon = 0.5$, whereas for $\varepsilon \leqslant 0.2$, the errors should be very much smaller. Since the error estimate is $O(\varepsilon^5)$ in rotation problems compared with $O(\varepsilon^3)$ in translation situations, even better accuracy should obtain for torques.

Returning to the question of exact solutions (in the numerical sense), free from the restriction $a/h \ll 1$, a number of configurations offer themselves for investigation. In the case of a sphere either translating along or rotating about an axis parallel to the surface, a representation of the velocity field due to Dean and O'Neill [15] can be used in conjunction with a bispherical coordinate system to effect a solution. When the body is a thin disk translating in its own plane which is parallel to the surface (wherein the Φ-factor of Section 3 is $16/9\pi$), we can represent the velocity field in terms of harmonic functions, and the solution of the resulting mixed boundary-value problems reduces to that of a system of coupled Fredholm integral equations of the second kind after the manner of [4]. The detailed analysis of these problems, together with some extensions in which a bulk fluid occupies the region $z < 0$, will be given in future papers.

Appendix

In this Appendix we collect together the values of various integrals which appear in the Hankel inversions of Sections 3 and 4. With $R=(\rho^2+h^2)^{1/2}$, Gradshteyn and Ryzhik ([16], p. 682) give the integral

$$\int_0^\infty \frac{\rho J_0(s\rho)}{R^3}\,\mathrm{d}\rho=\frac{1}{h}\,\mathrm{e}^{-sh}. \tag{A1}$$

Successive application of the operator $-h^{-1}\partial/\partial h$ then produces the results

$$\int_0^\infty \frac{\rho J_0(s\rho)}{R^5}\,\mathrm{d}\rho=\frac{1}{3h^3}(1+hs)\,\mathrm{e}^{-hs}, \tag{A2}$$

$$\int_0^\infty \frac{\rho J_0(s\rho)}{R^7}\,\mathrm{d}\rho=\frac{1}{5h^3}\left(1+hs+\tfrac{1}{3}h^2s^2\right)\mathrm{e}^{-hs}. \tag{A3}$$

Two further integrals obtainable from [16] are

$$\int_0^\infty \frac{J_1(s\rho)}{R}\,\mathrm{d}\rho=\frac{1}{hs}(1-\mathrm{e}^{-hs}), \tag{A4}$$

and

$$\int_0^\infty \frac{\rho^2 J_1(s\rho)}{R^3}\,\mathrm{d}\rho=\mathrm{e}^{-hs}. \tag{A5}$$

Repeated differentiation of (A4) and (A5) with respect to h now shows that

$$\int_0^\infty \frac{J_1(s\rho)}{R^3}\,\mathrm{d}\rho=\frac{1}{h^3s}\{1-(1+hs)\,\mathrm{e}^{-hs}\}, \tag{A6}$$

$$\int_0^\infty \frac{\rho^2 J_1(s\rho)}{R^5}\,\mathrm{d}\rho=\frac{s}{3h}\mathrm{e}^{-sh}, \tag{A7}$$

$$\int_0^\infty \frac{J_1(s\rho)}{R^5}\,\mathrm{d}\rho=\frac{1}{h^5s}\left\{1-\left(1+hs+\tfrac{1}{3}h^2s^2\right)\mathrm{e}^{-hs}\right\}, \tag{A8}$$

$$\int_0^\infty \frac{\rho^2 J_1(s\rho)}{R^7}\,\mathrm{d}\rho=\frac{s}{15h^3}(1+hs)\,\mathrm{e}^{-hs}, \tag{A9}$$

and

$$\int_0^\infty \frac{J_1(s\rho)}{R^7}\,\mathrm{d}\rho=\frac{1}{h^7s}\left\{1-\left(1+hs+\tfrac{2}{5}h^2s^2+\tfrac{1}{15}h^3s^3\right)\mathrm{e}^{-hs}\right\}. \tag{A10}$$

Finally, if in (A1) to (A3) we write

$$\rho J_0(s\rho)=\frac{2}{s}J_1(s\rho)-\rho J_2(s\rho),$$

and use (A6), (A8) and (A10), it is found that

$$\int_0^\infty \frac{\rho J_2(s\rho)}{R^3}\,d\rho = \frac{1}{h^3 s^2}\{2 - (2 + 2hs + h^2 s^2)\,e^{-hs}\}, \tag{A11}$$

$$\int_0^\infty \frac{\rho J_2(s\rho)}{R^5}\,d\rho = \frac{1}{h^5 s^2}\{2 - (2 + 2hs + h^2 s^2 + \tfrac{1}{3}h^3 s^3)\,e^{-hs}\}, \tag{A12}$$

and

$$\int_0^\infty \frac{\rho J_2(s\rho)}{R^7}\,d\rho = \frac{1}{h^7 s}\{2 - (2 + 2hs + h^2 s^2 + \tfrac{1}{3}h^3 s^3 + \tfrac{1}{15}h^4 s^4)\,e^{-hs}\}. \tag{A13}$$

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